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Travelling wave solutions to nonlinear evolution and wave equations

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Abstract. We have studied a series of (ansätze) ordinary differential equations of the first order, which correspond to the travelling (and/or solitary) wave solutions of some nonlinear partial differential equations. We have investigated the conditions, under which the nonlinear partial differential equations have certain kinds of travelling (and/or solitary) wave solutions. As a consequence of applications, we can take the trial procedures to obtain the travelling wave solutions, which is a very efficient method for solving several classes of nonlinear partial differential equations.

1. Introduction

Nonlinear evolution and wave equations are special classes of partial differential equations, which have been studied intensively for the past decades (e.g. see two recent books [1, 2]). When a nonlinear partial differential equation is used to describe a physical parameter which shows some kinds of propagation or aggregation properties, one of the important physical motivations is to solve the partial differential equation with the travelling (and/or solitary) solutions. However, due to the complexity of the mathematics, there are few exact travelling solutions obtained by limited techniques [1–14].

As far as the travelling solutions are concerned, one can always use the transform:

$$\xi = x - ct$$

thus, the nonlinear partial differential equation will be simplified to a nonlinear ordinary differential equation. However, solving the nonlinear ordinary differential equation is also a very difficult target to achieve.

Hereman and co-workers [9] recently introduced an algebra approach to obtaining the travelling (and/or solitary) wave solutions. Their method is straightforward, but has difficulty in summing a series of expansion related to the recursion relations of the coefficients.

Using a special nonlinear transform Wang *et al* [11] obtained a travelling solution to a generalized Fisher equation. Assuming the solution to be a polynomial of hyperbolic tangent functions, Lan and Wang [12] obtained the exact solutions for two nonlinear evolution equations. However, their approach is restricted for the special problem. It is difficult to extend their method to more general nonlinear evolution/wave equations.

Quite recently, Lu and co-workers [13] introduced the Bernoulli equation as an ansatz to solve some nonlinear diffusion equations. Through simple algebraic calculations, they obtained the travelling wave solutions to the Newell–Whitehead equation, the generalized Burgers–Fisher equation, and the generalized Burgers–Huxley equation, etc. Inspired by Lu *et al*'s work, we have studied a series of new ansätze, which can be related to some nonlinear evolution/wave equations in a more general sense. Applying this approach, we have obtained several new travelling (and/or solitary) wave solutions to some generalized Korteweg–de Vries (KdV) equations, generalized Burgers equation, and modified sine–Gordon equations, etc. We report these results in this paper.

This paper is arranged as follows. We present a series of ansätze whose solutions correspond to the general nonlinear evolution and wave equations in section 2. The examples presented in section 3 show how to obtain the travelling (and/or solitary) wave solutions through trivial algebraic operations. Section 4 is a brief discussion and outlook about the further extension to the study of nonlinear evolution/wave equations for high-dimensional systems and a coupled nonlinear evolution/wave equation system.

2. Ansätze and solutions

Let us consider the following nonlinear partial differential (evolution/wave) equations:

$$\begin{aligned} f_1(u)u_t + f_2(u)u_x + f_3(u)u_{tt} + f_4(u)u_{tx} + f_5(u)u_{xx} + f_6(u)u_t^2 + f_7(u)u_tu_x + f_8(u)u_x^2 \\ +f_9(u)u_{ttt} + f_{10}(u)u_{ttx} + f_{11}(u)u_{txx} + f_{12}(u)u_{xxx} + f_{13}(u)u_tu_{tt} \\ +f_{14}(u)u_{tt}u_x + f_{15}(u)u_tu_{xx} + f_{16}(u)u_xu_{xx} + f_{17}(u)u_t^3 \\ +f_{18}(u)u_t^2u_x + f_{19}(u)u_tu_x^2 + f_{20}(u)u_x^3 + \cdots = g(u) \end{aligned} \tag{1}$$

where $f_i(u)$ $(i = 1, 2, 3, \ldots)$ and $g(u)$ are algebraic functions of u (such as polynomials, rational functions, and triangle functions etc), $u_t = \partial u/\partial t$, and $u_x = \partial u/\partial x$, etc. Since we are only interested in the travelling (and/or solitary) wave solutions, we let

$$\xi = x - ct \tag{2}$$

where c is the speed of the propagating waves. Equation (1) is thus transformed to a nonlinear ordinary differential equation as

$$\begin{aligned} [-cf_1 + f_2]u' + [c^2f_3 - cf_4 + f_5]u'' + [c^2f_6 - cf_7 + f_8](u')^2 \\ +[-c^3f_9 + c^2f_{10} - cf_{11} + f_{12}]u''' \\ +[-c^3f_{13} + c^2f_{14} - cf_{15} + f_{16}]u'u'' \\ +[-c^3f_{17} + c^2f_{18} - cf_{19} + f_{20}](u')^3 + \cdots = g(u)\,. \end{aligned} \tag{3}$$

In fact, it is almost impossible to solve this nonlinear ordinary differential equation for a general consideration. We now look for some typical travelling (and/or solitary) wave solutions which are well defined.

2.1. Case 1

For a general consideration of hyperbolic cosecant function solutions, we consider the ansatz

$$u' = -\nu bu\sqrt{1 - \left(\frac{u}{a}\right)^{2/\nu}} \tag{4}$$

where ν, a, and b are real numbers, and $\nu > 0$. Through integration, we obtain the solution as

$$u(\xi) = a\ \mathrm{sech}^{\nu}(b\xi + c_0)\,. \tag{5}$$

Thus, we have

$$u'' = \nu^2 b^2 u\left[1-\left(1+\frac{1}{\nu}\right)\left(\frac{u}{a}\right)^{2/\nu}\right] = -\frac{\nu b}{\sqrt{1-(u/a)^{2/\nu}}}\left[1-\left(1+\frac{1}{\nu}\right)\left(\frac{u}{a}\right)^{2/\nu}\right]u'$$

$$u''' = \nu^2 b^2\left[1-\frac{(\nu+1)(\nu+2)}{\nu^2}\left(\frac{u}{a}\right)^{2/\nu}\right]u'$$

$$u^{(4)} = \frac{-b^3}{\sqrt{1-(u/a)^{2/\nu}}}\left[\nu^3-2(\nu+1)(2+2\nu+\nu^2)\left(\frac{u}{a}\right)^{2/\nu}\right.$$
$$\left.+(\nu+1)(\nu+2)(\nu+3)\left(\frac{u}{a}\right)^{4/\nu}\right]u'$$

$$u^{(5)} = b^4\left[\nu^4-2(\nu+1)(\nu+2)(2+2\nu+\nu^2)\left(\frac{u}{a}\right)^{2/\nu}\right.$$
$$\left.+(\nu+1)(\nu+2)(\nu+3)(\nu+4)\left(\frac{u}{a}\right)^{4/\nu}\right]u'$$

$$\vdots \qquad \vdots\,.$$

Substituting these results into (3) yields

$$\left\{f_2 - cf_1 - \frac{\nu b(c^2 f_3 - cf_4 + f_5)}{\sqrt{1-(u/a)^{2/\nu}}}\left[1-\left(1+\frac{1}{\nu}\right)\left(\frac{u}{a}\right)^{2/\nu}\right]\right.$$
$$\left. -\nu bu\sqrt{1-(u/a)^{2/\nu}}\,(c^2 f_6 - cf_7 + f_8) + \cdots\right\}\nu bu\sqrt{1-\left(\frac{u}{a}\right)^{2/\nu}} \equiv -g(u)\,. \tag{6}$$

2.2. *Case 2*

Similarly, for a general consideration of hyperbolic tangent function solution, we consider the following ansatz:

$$u' = \nu bu\left(\frac{a}{u}\right)^{1/\nu}\left[1-\left(\frac{u}{a}\right)^{2/\nu}\right] \tag{7}$$

where a, b, and ν are real numbers, and $\nu > 0$. The solution is

$$u(\xi) = a\ \tanh^{\nu}(b\xi + c_0)\,. \tag{8}$$

This solution can be considered as a special case of the trial solutions suggested by Lan and Wang [12]. Thus, we have

$$u'' = \nu b^2 u\left[(\nu-1)\left(\frac{a}{u}\right)^{2/\nu} - 2\nu + (\nu+1)\left(\frac{u}{a}\right)^{2/\nu}\right]$$
$$= b\left(\frac{a}{u}\right)^{1/\nu}\left[\nu - 1 - (\nu+1)\left(\frac{u}{a}\right)^{2/\nu}\right]u'$$

$$u''' = b^2\left[(\nu-1)(\nu-2)\left(\frac{a}{u}\right)^{2/\nu} - 2\nu^2 + (\nu+1)(\nu+2)\left(\frac{u}{a}\right)^{2/\nu}\right]u'$$

$$u^{(4)} = b^3\left[(\nu-1)(\nu-2)(\nu-3)\left(\frac{a}{u}\right)^{3/\nu} - (\nu-1)(2-3\nu+3\nu^2)\left(\frac{a}{u}\right)^{1/\nu} \right.$$
$$\left. +(\nu+1)(2+3\nu+3\nu^2)\left(\frac{u}{a}\right)^{1/\nu} - (\nu+1)(\nu+2)(\nu+3)\left(\frac{u}{a}\right)^{3/\nu}\right]u'$$

$\vdots \qquad \vdots .$

Substituting these results into (3) yields

$$\nu bu\left\{f_2 - cf_1 + b\left(\frac{a}{u}\right)^{1/\nu}\left[\nu - 1 - (\nu+1)\left(\frac{u}{a}\right)^{2/\nu}\right](c^2 f_3 - cf_4 + f_5)\right.$$
$$\left. + \nu bu\left(\frac{a}{u}\right)^{1/\nu}\left[1 - \left(\frac{u}{a}\right)^{2/\nu}\right](c^2 f_6 - cf_7 + f_8) + \cdots\right\}$$
$$\times\left(\frac{a}{u}\right)^{1/\nu}\left[1 - \left(\frac{u}{a}\right)^{2/\nu}\right] \equiv g(u). \tag{9}$$

2.3. Case 3

For the ansatz (Bernoulli equation)

$$u' = au + bu^n \tag{10}$$

where a, b, and n are real numbers, $ab < 0$ (here, this condition can be extended to the case $ab > 0$ for some special case, e.g. see example 9 below), $n \neq 1$, and its solution

$$u(\xi) = \left[-\frac{a}{2b}\tanh\left(\frac{n-1}{2}a\xi + c_0\right) - \frac{a}{2b}\right]^{1/(n-1)} \tag{11}$$

have been introduced to study a particular diffusion equation by Lu *et al* [13]. We now present a more general result.

From the ansatz, we have

$$u'' = (a + bnu^{n-1})u' = a^2 + ab(n+1)n^{n-1} + b^2nu^{2n-2}$$
$$u''' = [a^2 + abn(n+1)u^{n-1} + b^2n(2n-1)u^{2n-2}]u'$$

$\vdots \qquad \vdots .$

Thus, equation (11) is a solution to (3), if and only if the coefficient functions satisfy the relationship

$$[f_2 - cf_1 + (a + bnu^{n-1})(c^2 f_3 - cf_4 + f_5)$$
$$+(au + bu^n)(c^2 f_6 - cf_7 + f_8) + \cdots](au + bu^n) \equiv g(u). \tag{12}$$

2.4. Case 4

We introduce the ansatz

$$u' = a_0 + a_1 u + a_2 u^2 \tag{13}$$

i.e. the Riccati equation with real constant coefficients. When $a_0 = 0$, it is a special case of (10) for $n = 2$, and was introduced by Wang *et al* [11]. However, it is distinguished from them when $a_0 \neq 0$. Its solutions can be expressed as

$$u(\xi) = \frac{\sqrt{\Delta}}{2a_2}\tan\left(\frac{\sqrt{\Delta}}{2}\xi + c_0\right) - \frac{a_1}{2a_2} \tag{14}$$

for $\Delta = 4a_0a_2 - a_1^2 > 0$, and

$$u(\xi) = -\frac{\sqrt{-\Delta}}{2a_2} \tanh\left(\frac{\sqrt{-\Delta}}{2}\xi + c_0\right) - \frac{a_1}{2a_2} \tag{15}$$

for $\Delta = 4a_0a_2 - a_1^2 < 0$. Since we have

$$\begin{aligned} u'' &= (a_1 + 2a_2u)\ u' \\ u''' &= (2a_0a_2 + a_1^2 + 6a_1a_2u + 6a_2^2u^2)\ u' \\ \vdots \quad & \quad \vdots \end{aligned}$$

the coefficient functions need to satisfy the relationship

$$[f_2 - cf_1 + (a_1 + 2a_2u)(c^2f_3 - cf_4 + f_5) + (a_0 + a_1u + a_2u^2)(c^2f_6 - cf_7 + f_8) + \cdots] \times(a_0 + a_1u + a_2u^2) \equiv g(u)\,. \tag{16}$$

2.5. Case 5

For a general consideration of triangle function solutions, we consider the ansatz

$$u' = \nu bu\sqrt{\left(\frac{a}{u}\right)^{2/\nu} - 1} \tag{17}$$

where a, b, and ν are real numbers, and $\nu > 0$. Through integration, we obtain the solution as

$$u(\xi) = a\ \sin^\nu(b\xi + c_0)\,. \tag{18}$$

Thus, we have

$$\begin{aligned} u'' &= -\nu^2b^2u\left[1 - \left(1 - \frac{1}{\nu}\right)\left(\frac{a}{u}\right)^{2/\nu}\right] \\ &= -\frac{\nu b}{\sqrt{(a/u)^{2/\nu} - 1}}\left[1 - \left(1 - \frac{1}{\nu}\right)\left(\frac{a}{u}\right)^{2/\nu}\right] u' \end{aligned}$$

$$\begin{aligned} u''' &= -\nu^2b^2\left[1 - \frac{(\nu-1)(\nu-2)}{\nu^2}\left(\frac{a}{u}\right)^{2/\nu}\right] u' \\ \vdots \quad & \quad \vdots\,. \end{aligned}$$

Substituting these results into (3) yields

$$\left\{f_2 - cf_1 - \frac{\nu b(c^2f_3 - cf_4 + f_5)}{\sqrt{(u/a)^{2/\nu} - 1}}\left[1 - \left(1 - \frac{1}{\nu}\right)\left(\frac{u}{a}\right)^{2/\nu}\right] + \nu bu\sqrt{\left(\frac{a}{u}\right)^{2/\nu} - 1}\ (c^2f_6 - cf_7 + f_8) + \cdots\right\}\nu bu\sqrt{\left(\frac{u}{a}\right)^{2/\nu} - 1} \equiv g(u)\,. \tag{19}$$

2.6. Case 6

We now consider an ansatz related to the hyperbolic cosecant function solution:

$$u' = -\nu b u \sqrt{1 + \left(\frac{u}{a}\right)^{2/\nu}} \tag{20}$$

where ν, a, and b are real numbers. Through integration, we obtain the solution as

$$u(\xi) = a \operatorname{cosech}^{\nu}(b\xi + c_0). \tag{21}$$

Thus, we have

$$\begin{aligned} u'' &= \nu^2 b^2 u \left[1 + \frac{\nu+1}{\nu}\left(\frac{u}{a}\right)^{2/\nu}\right] \\ &= -\frac{\nu b}{\sqrt{1+(u/a)^{2/\nu}}}\left[1 + \frac{\nu+1}{\nu}\left(\frac{u}{a}\right)^{2/\nu}\right] u' \\ u''' &= \nu^2 b^2 \left[1 + \frac{(\nu+1)(\nu+2)}{\nu^2}\left(\frac{u}{a}\right)^{2/\nu}\right] u' \\ &\vdots \qquad \vdots . \end{aligned}$$

Substituting these results into (3) yields

$$\left\{ f_2 - c f_1 - \frac{\nu b (c^2 f_3 - c f_4 + f_5)}{\sqrt{1+(u/a)^{2/\nu}}}\left[1 + \frac{\nu+1}{\nu}\left(\frac{u}{a}\right)^{2/\nu}\right] \right. \\ \left. - \nu b u \sqrt{1 + \left(\frac{u}{a}\right)^{2/\nu}}\,(c^2 f_6 - c f_7 + f_8) + \cdots \right\} \nu b u \sqrt{1 + \left(\frac{u}{a}\right)^{2/\nu}} \equiv -g(u). \tag{22}$$

2.7. Case 7

Consider the following ansatz:

$$u' = \frac{\nu b u}{2}\left(\frac{a}{u}\right)^{1/\nu} \sin\left[2\left(\frac{u}{a}\right)^{1/\nu}\right] \tag{23}$$

where ν, a, and b are real numbers. Through integration, we obtain the solution as

$$u(\xi) = a \tan^{-1\,\nu}[\exp(b\xi + c_0)]. \tag{24}$$

Thus, we have

$$\begin{aligned} u'' &= b\left\{ \cos\left[2\left(\frac{u}{a}\right)^{1/\nu}\right] + \frac{(\nu-1)}{2}\left(\frac{a}{u}\right)^{1/\nu} \sin\left[2\left(\frac{u}{a}\right)^{1/\nu}\right]\right\} u' \\ u''' &= \frac{b^2}{8}\left(\frac{a}{u}\right)^{2/\nu} \left\{ (\nu-1)(\nu-2) - (\nu-1)(\nu-2)\cos\left[4\left(\frac{u}{a}\right)^{1/\nu}\right] \right. \\ &\quad \left. + 6(\nu-1)\left(\frac{u}{a}\right)^{1/\nu} \sin\left[4\left(\frac{u}{a}\right)^{1/\nu}\right] + 8\left(\frac{u}{a}\right)^{2/\nu} \cos\left[4\left(\frac{u}{a}\right)^{1/\nu}\right]\right\} u' \\ &\vdots \qquad \vdots . \end{aligned}$$

Substituting these results into (3) yields

$$\left\{f_2 - cf_1 + (c^2 f_3 - cf_4 + f_5)b\left(\cos\left[2\left(\frac{u}{a}\right)^{1/\nu}\right] + \frac{(\nu-1)}{2}\left(\frac{a}{u}\right)^{1/\nu}\sin\left[2\left(\frac{u}{a}\right)^{1/\nu}\right]\right)\right.$$
$$\left. +(c^2 f_6 - cf_7 + f_8)\frac{\nu bu}{2}\left(\frac{a}{u}\right)^{1/\nu}\sin\left[2\left(\frac{u}{a}\right)^{1/\nu}\right] + \cdots\right\}$$
$$\times\frac{\nu bu}{2}\left(\frac{a}{u}\right)^{1/\nu}\sin\left[2\left(\frac{u}{a}\right)^{1/\nu}\right] \equiv g(u)\,. \tag{25}$$

2.8. Case 8

We consider the following ansatz:

$$u' = -\nu bu\left(\frac{a}{u}\right)^{1/\nu}\sin^2\left[\left(\frac{u}{a}\right)^{1/\nu}\right] \tag{26}$$

where ν, a, and b are real numbers. Through integration, we obtain the solution as

$$u(\xi) = a\cot^{-1\,\nu}(b\xi + c_0)\,. \tag{27}$$

Thus, we have

$$u'' = b^2\nu u\left(\frac{a}{u}\right)^{2/\nu}\sin^3\left[\left(\frac{u}{a}\right)^{1/\nu}\right]\left\{2\left(\frac{u}{a}\right)^{1/\nu}\cos\left[\left(\frac{u}{a}\right)^{1/\nu}\right] + (\nu-1)\sin\left[\left(\frac{u}{a}\right)^{1/\nu}\right]\right\}$$
$$= -b\sin\left[\left(\frac{u}{a}\right)^{1/\nu}\right]\left\{2\cos\left[\left(\frac{u}{a}\right)^{1/\nu}\right] + (\nu-1)\left(\frac{a}{u}\right)^{1/\nu}\sin\left[\left(\frac{u}{a}\right)^{1/\nu}\right]\right\}u'$$

$$u''' = \frac{b^2}{2}\left(\frac{a}{u}\right)^{2/\nu}\sin^2\left[\left(\frac{u}{a}\right)^{1/\nu}\right]\left\{(\nu-1)(\nu-2) + 4\left(\frac{u}{a}\right)^{2/\nu}\right.$$
$$-(\nu-1)(\nu-2)\cos\left[2\left(\frac{u}{a}\right)^{1/\nu}\right] + 8\left(\frac{u}{a}\right)^{2/\nu}\cos\left[2\left(\frac{u}{a}\right)^{1/\nu}\right]$$
$$\left. +6(\nu-1)\left(\frac{u}{a}\right)^{1/\nu}\sin\left[2\left(\frac{u}{a}\right)^{1/\nu}\right]\right\}u'$$

$$\vdots \qquad \vdots\,.$$

Substituting these results into (3) yields

$$\left\{f_2 - cf_1 - b(c^2 f_3 - cf_4 + f_5)\left(\frac{a}{u}\right)^{1/\nu}\sin\left[\left(\frac{u}{a}\right)^{1/\nu}\right]\right.$$
$$\times\left(2\left(\frac{u}{a}\right)^{1/\nu}\cos\left[\left(\frac{u}{a}\right)^{1/\nu}\right] + (\nu-1)\sin\left[\left(\frac{u}{a}\right)^{1/\nu}\right]\right)$$
$$\left. -\nu bu(c^2 f_6 - cf_7 + f_8)\left(\frac{a}{u}\right)^{1/\nu}\sin^2\left[\left(\frac{u}{a}\right)^{1/\nu}\right] + \cdots\right\}$$
$$\times\nu bu\left(\frac{a}{u}\right)^{1/\nu}\sin^2\left[\left(\frac{u}{a}\right)^{1/\nu}\right] \equiv -g(u)\,. \tag{28}$$

2.9. Case 9

The following is also a triangle function ansatz:

$$u' = -\frac{ab\nu}{2}\sin\left(\frac{2u}{a}\right)\left(\tan\frac{u}{a}\right)^{1/\nu} \tag{29}$$

where a, b and ν are real numbers. Its solitary solution can be obtained as

$$u(\xi) = a\cot^{-1}(b\xi + c_0)^{\nu}. \tag{30}$$

Thus, we have

$$u'' = -b\left[1 + \nu\cos\left(\frac{2u}{a}\right)\right]\tan^{1/\nu}\left(\frac{u}{a}\right)u'$$
$$u''' = b^2\left[2 + 3\nu\cos\left(\frac{2u}{a}\right) + \nu^2\cos\left(\frac{4u}{a}\right)\right]\tan^{2/\nu}\left(\frac{u}{a}\right)u'$$
$$\vdots \qquad \vdots.$$

The coefficient functions satisfy the relationship

$$\Big[f_2 - cf_1 - (c^2 f_3 - cf_4 + f_5)b\left[1 + \nu\cos\left(\frac{2u}{a}\right)\right]\tan^{1/\nu}\left(\frac{u}{a}\right) - (c^2 f_6 - cf_7 + f_8)\frac{ab\nu}{2}\sin\left(\frac{2u}{a}\right)\left(\tan\frac{u}{a}\right)^{1/\nu} + \cdots\Big] \times\frac{ab\nu}{2}\sin\left(\frac{2u}{a}\right)\left(\tan\frac{u}{a}\right)^{1/\nu} \equiv -g(u). \tag{31}$$

3. Examples

We now will use the trial procedure to obtain the travelling wave solutions to some nonlinear evolution/wave equations.

Example 1. For the first example, we study a generalized KdV equation

$$u_t + \beta u^{\alpha}u_x + \gamma u_{xxx} = 0 \tag{32}$$

where α and β are positive real numbers. This equation reduces to the original KdV equation and a modified KdV equation for $\alpha = 1, 2$, respectively. Verheest [4] derived the gKdV equation with $\alpha = 3$ for describing the propagation of ion-acoustic waves at critical densities in a multi-component plasma with different ionic charges and temperatures. Schamel [5] derived the gKdV equation with $\beta = 1$ and $\alpha = \frac{1}{2}$ for describing ion-acoustic waves in a cold-ion plasma but where the electrons do not behave isothermally during their passage of the wave. To our knowledge, there is no report about the general case for $\alpha > 0$.

Since it is a homogeneous equation and its coefficients are polynomials of u, first, we consider that case 1 may be applicable to this problem. Suppose that its solution can be expressed in the form of (5), some algebra calculation yields the solitary solution

$$u(\xi) = u(x - ct) = \left[\frac{c(\alpha+1)(\alpha+2)}{2\beta}\right]^{1/\alpha}\operatorname{sech}^{2/\alpha}\left[\pm\frac{\alpha}{2}\sqrt{\frac{c}{\gamma}}(x - ct) + c_0\right]. \tag{33}$$

It requires $c/\gamma > 0$. It is clear that this solution is consistent with the results for $\alpha = 1$ and 2 [9].

Second, we consider case 2 may be applicable for $\alpha = 1, 2$. So that the solution can be expressed in the form of (8). We now determine the parameters, a, b, and ν, by comparing coefficients. Thus, (9) can be expressed explicitly as

$$\beta u^\alpha - c + \gamma b^2\left[(\nu-1)(\nu-2)\left(\frac{a}{u}\right)^{2/\nu} - 2\nu^2 + (\nu+1)(\nu+2)\left(\frac{u}{a}\right)^{2/\nu}\right] \equiv 0\,. \tag{34}$$

This equation yields the equation system for $\nu = 2/\alpha = 1, 2$:

$$c + 2\gamma b^2\nu^2 = 0 \qquad \text{and} \qquad \beta - (\nu+1)(\nu+2)\frac{c}{2a^{2/\nu}\nu^2} = 0\,.$$

The solutions are

$$a = \left[\frac{c}{4\beta}(\alpha+1)(\alpha+2)\right]^{1/\alpha} \qquad \text{and} \qquad b = \pm\alpha\sqrt{\frac{-c}{8\gamma}}\,.$$

It requires $c/\gamma < 0$ when a and b are real numbers.

Example 2. The second example is another generalized KdV equation:

$$u_t + \beta u^\alpha u_x + \gamma u^\tau u_x u_{xx} + \delta u_{xxx} = 0 \tag{35}$$

where α, β and τ are real numbers.

First, we consider α, $\tau > 0$, and try to use the solution in case 1. We assume the solution can be expressed in the form of (5). Thus, substituting it into (6) yields

$$\beta u^\alpha - c + \gamma b^2\nu^2 u^{\tau+1}\left[1 - \frac{\nu+1}{\nu}\left(\frac{u}{a}\right)^{2/\nu}\right] + \delta b^2\nu^2\left[1 - \frac{(\nu+1)(\nu+2)}{\nu^2}\left(\frac{u}{a}\right)^{2/\nu}\right] \equiv 0\,. \tag{36}$$

If β, γ, δ, ν, $c > 0$, this equality requires

$$\alpha = \tau + 1 + 2/\nu \qquad \text{and} \qquad \tau + 1 = 2/\nu$$

and

$$\delta b^2\nu^2 - c = 0 \qquad \beta - \gamma b^2 a^{-2/\nu}\nu(\nu+1) = 0 \qquad \gamma b^2\nu^2 - \delta b^2 a^{-2/\nu}(\nu+1)(\nu+2) = 0\,.$$

Under the condition $\alpha = 2(\tau+1)$ the solutions are

$$\nu = 2/(\tau+1) \qquad a = \left[(\tau+2)(\tau+3)\frac{\delta}{2\gamma}\right]^{1/(\tau+1)}$$

$$b = \pm\frac{\tau+1}{2\gamma}\sqrt{\beta\delta(\tau+2)} \qquad c = \beta(\tau+2)\frac{\delta^2}{\gamma^2}\,.$$

We now study the solutions matching case 2. Thus, the solution may be expressed in the form of (8). Substituting it into (9) yields

$$\beta u^\alpha - c + \gamma\nu b^2 u^{\tau+1}\left[(\nu-1)\left(\frac{a}{u}\right)^{2/\nu} - 2\nu + (\nu+1)\left(\frac{u}{a}\right)^{2/\nu}\right]$$
$$+\delta b^2\left[(\nu-1)(\nu-2)\left(\frac{a}{u}\right)^{2/\nu} - 2\nu^2 + (\nu+1)(\nu+2)\left(\frac{u}{a}\right)^{2/\nu}\right] \equiv 0\,. \tag{37}$$

In the following, we consider three cases: (1) $\nu = 1$, (2) $\nu = 2$, and (3) $\nu \neq 1$, $\nu \neq 2$.
(1) $\nu = 1$. Thus, the $(a/u)^2$terms vanish. The equality requires

$$\tau = 1 \qquad \text{and} \qquad \alpha = 4\,.$$

In this case, for $\beta < 0$ and δ, $\gamma > 0$, we have the solutions

$$a = \pm\sqrt{\frac{3\delta}{\gamma}} \qquad b = \pm\sqrt{\frac{-3\beta\delta}{2\gamma^2}} \qquad c = 3\beta\frac{\delta^2}{\gamma^2} < 0\,.$$

(2) $\nu = 2$. For $\alpha = 2$, $\tau = 0$, and $\beta\delta < 0$, we have the solutions

$$a = \frac{3\delta}{2\gamma} \qquad b = \pm\sqrt{\frac{-\beta\delta}{4\gamma^2}} \qquad c = \frac{5\beta\delta^2}{4\gamma^2}\,.$$

(3) $\nu \neq 1$, $\nu \neq 2$.
(3*a*) There is no solution for α, $\tau \geqslant 0$.
(3*b*) For $\alpha > 0$ and $c > 0$, if and only if

$$\tau = -1$$
$$\delta < 0 \quad \gamma > 0 \qquad \text{and} \qquad \delta + \gamma < 0$$
$$\alpha = 2/\nu = (\delta + \gamma)/\delta > 0$$

the equality can be true. In this case, the solutions are

$$\nu = \frac{2}{\alpha} = \frac{2\delta}{\delta + \gamma} \qquad a = \left[\frac{c(3\delta + \gamma)}{2\beta\delta}\right]^{1/\alpha}$$
$$b = \pm\sqrt{\frac{-c\alpha^2}{8(\delta + \gamma)}} = \pm\sqrt{\frac{-c(\delta + \gamma)}{8\delta^2}}\,.$$

Example 3. Let us consider a generalized Benjamin–Bona–Mahony equation:

$$u_t + (\beta u^\alpha + 1)u_x - \gamma u_{txx} = 0 \tag{38}$$

where α, β and γ are positive real numbers. When $\alpha = 1$ and $\gamma = 1$, it reduces to the original Benjamin–Bona–Mahony [3] equation.

We consider applying case 1 to this equation, i.e. assuming the travelling solitary solution can be expressed in the form of (5). Substituting the coefficient functions into (6) yields

$$\beta u^\alpha + 1 - c + c\gamma b^2\nu^2\left[1 - \frac{(\nu + 1)(\nu + 2)}{\nu^2}\left(\frac{u}{a}\right)^{2/\nu}\right] \equiv 0\,. \tag{39}$$

Letting $\nu = 2/\alpha$, we obtain

$$a = \left[\frac{(c - 1)(\alpha + 1)(\alpha + 2)}{2\beta}\right]^{1/\alpha} \qquad \text{and} \qquad b = \pm\frac{\alpha}{2}\sqrt{\frac{c - 1}{c\gamma}}\,.$$

That a and b are real numbers requires $c > 1$.

Example 4. Let us consider a generalized Joseph–Egri equation:

$$u_t + (\beta u^\alpha + 1)u_x + \gamma u_{ttx} = 0 \tag{40}$$

where α, β and γ are real numbers, and α, $\gamma > 0$. When $\alpha = \gamma = 1$ and $\beta > 0$, it reduces to the original Joseph–Egri equation.

When $\beta > 0$, we consider applying case 1 to this equation, i.e. assuming the travelling solitary solution can be expressed in the form of (5). Substituting the coefficient functions into (6) yields

$$\beta u^\alpha + 1 - c + c^2\gamma b^2\nu^2\left[1 - \frac{(\nu + 1)(\nu + 2)}{\nu^2}\left(\frac{u}{a}\right)^{2/\nu}\right] \equiv 0\,. \tag{41}$$

Letting $\nu = 2/\alpha$, we obtain

$$a = \left[\frac{(c-1)(\alpha+1)(\alpha+2)}{2\beta}\right]^{1/\alpha} \qquad \text{and} \qquad b = \pm\frac{\alpha}{2c}\sqrt{\frac{c-1}{\gamma}}\,.$$

When $\beta < 0$, we apply case 6, and obtain the solution as

$$u(x-ct) = \left[\frac{(c-1)(\alpha+1)(\alpha+2)}{-2\beta}\right]^{1/\alpha} \operatorname{cosech}^{2/\alpha}\left[\pm\frac{\alpha}{2c}\sqrt{\frac{c-1}{\gamma}}(x-ct)+c_0\right]. \tag{42}$$

For both cases, that a and b are real numbers requires $c > 1$.

Example 5. Let us consider a generalized fifth-order KdV equation, which reads

$$u_t + \beta u^\alpha u_x + \gamma u_{xxx} + \delta u_{xxxxx} = 0 \tag{43}$$

where $\alpha > 0$, β, $\gamma < 0$ and δ are real numbers. We apply case 1 to this equation, i.e. assuming the solution can be expressed by (5), substituting it into (6) yields

$$\begin{aligned}\beta u^\alpha - c + \gamma b^2\nu^2&\left[1 - \frac{(\nu+1)(\nu+2)}{\nu^2}\left(\frac{u}{a}\right)^{2/\nu}\right] \\ &+\delta b^4\left[\nu^4 - 2(\nu+1)(\nu+2)(2+2\nu+\nu^2)\left(\frac{u}{a}\right)^{2/\nu}\right. \\ &\left.+(\nu+1)(\nu+2)(\nu+3)(\nu+4)\left(\frac{u}{a}\right)^{4/\nu}\right] \cong 0\,.\end{aligned} \tag{44}$$

Letting $\nu = 4/\alpha$, we have the equation system:

$$\begin{aligned}&c - \gamma b^2\nu^2 - \delta b^4\nu^4 = 0 \\ &\gamma + 2\delta b^2(2+2\nu+\nu^2) = 0 \\ &\beta + \delta b^4(\nu+1)(\nu+2)(\nu+3)(\nu+4)a^{-\alpha} = 0\,.\end{aligned}$$

The solutions are

$$a = \left[\frac{-\gamma^2(\alpha+1)(\alpha+2)(3\alpha+4)(\alpha+4)}{2\beta\delta(8+4\alpha+\alpha^2)^2}\right]^{1/\alpha}$$

$$b = \pm\sqrt{\frac{-\alpha^2\gamma}{4\delta(8+4\alpha+\alpha^2)}} \qquad \text{and} \qquad c = \frac{-4\gamma^2(\alpha+2)^2}{\delta(8+4\alpha+\alpha^2)^2}\,.$$

Example 6. We now also consider another fifth-order KdV equation, which reads

$$u_t + \delta(1+\beta u^\alpha)u^\alpha u_x + \gamma u_{xxxxx} = 0 \tag{45}$$

where $\alpha, \gamma, \delta > 0$ and $\beta < 0$ are real numbers. Applying case 1 yields the equation

$$\begin{aligned}\delta(1+\beta u^\alpha)u^\alpha - c + \gamma b^4&\left[\nu^4 - 2(\nu+1)(\nu+2)(2+2\nu+\nu^2)\left(\frac{u}{a}\right)^{2/\nu}\right. \\ &\left.+(\nu+1)(\nu+2)(\nu+3)(\nu+4)\left(\frac{u}{a}\right)^{4/\nu}\right] \cong 0\,.\end{aligned} \tag{46}$$

Letting $\nu = 2/\alpha$, we have the equation system:

$$\begin{aligned}&c - \gamma b^4\nu^4 = 0 \\ &\delta - 2\gamma b^4(\nu+1)(\nu+2)(2+2\nu+\nu^2)a^{-\alpha} = 0 \\ &\beta\delta + \gamma b^4(\nu+1)(\nu+2)(\nu+3)(\nu+4)a^{-2\alpha} = 0\,.\end{aligned}$$

The solutions are

$$a = \left[\frac{-(2+3\alpha)(1+2\alpha)}{2\beta(2+2\alpha+\alpha^2)}\right]^{1/\alpha}$$
$$b = \pm\frac{\alpha}{2}\left[\frac{-\delta(2+3\alpha)(1+2\alpha)}{\beta\gamma(\alpha+1)(\alpha+2)(2+2\alpha+\alpha^2)^2}\right]^{1/4}$$
$$c = \frac{-\delta(2+3\alpha)(1+2\alpha)}{\beta(\alpha+1)(\alpha+2)(2+2\alpha+\alpha^2)^2}\,.$$

Example 7. We now consider a generalized Sharma–Tasso–Olver equation:

$$u_t + \beta u^\alpha u_x + \gamma u^\tau u_x^2 + \delta u^\sigma u_{xx} + \xi u_{xxx} = 0 \qquad (47)$$

where α, β, γ, δ, ξ, τ and σ are real numbers. When $\alpha = 2$, $\beta = \gamma = \delta = 3$, $\tau = 0$, $\sigma = 1$, and $\xi = 1$, it reduces to the original Sharma–Tasso–Olver [6] equation.

First, we consider applying case 2 to this equation, i.e. the solution is assumed to be in the form of (8). Substituting the coefficient functions into (9) yields

$$\beta u^\alpha - c + \gamma\nu b u^{\tau+1}\left[\left(\frac{a}{u}\right)^{1/\nu} - \left(\frac{u}{a}\right)^{1/\nu}\right] + \delta b u^\sigma\left[(\nu-1)\left(\frac{a}{u}\right)^{1/\nu} - (\nu+1)\left(\frac{u}{a}\right)^{1/\nu}\right]$$
$$+\xi b^2\left[(\nu-1)(\nu-2)\left(\frac{a}{u}\right)^{2/\nu} - 2\nu^2 + (\nu+1)(\nu+2)\left(\frac{u}{a}\right)^{2/\nu}\right] \equiv 0\,. \qquad (48)$$

For simplicity, we only consider three cases: (1) $\nu = 1$, (2) $\nu = 2$, and (3) $\nu \neq 1$, $\nu \neq 2$.

(1) $\nu = 1$. The equation of coefficient functions is simplified to be

$$\beta u^\alpha - c - 2\delta a^{-1} b u^{\sigma+1} + \gamma b u^{\tau+1}(a u^{-1} - a^{-1}u) - 2\xi b^2(1 - 3a^{-2}u^2) \equiv 0\,. \qquad (49)$$

In the following, we only discuss some typical cases.

(1*a*) $\alpha = 0$, $\sigma = 1$, $\tau = 0$. We have the equation system

$$\beta - c + \gamma ab - 2\xi b^2 = 0 \qquad 6\xi a^{-1}b - 2\delta - \gamma = 0\,.$$

The solutions are

$$a = \pm\sqrt{\frac{9\xi(\beta-c)}{(2\delta+\gamma)(\delta-\gamma)}} \qquad \text{and} \qquad b = \pm\sqrt{\frac{(\beta-c)(2\delta+\gamma)}{4\xi(\delta-\gamma)}}\,.$$

It requires that $(\beta - c)/(\delta - \gamma) > 0$ for a and b real numbers.

(1*b*) $\alpha = 1$, $\sigma = 0$, $\tau = 0$. We have the equation system:

$$\beta - 2\delta a^{-1}b = 0 \qquad 6\xi a^{-1}b - \gamma = 0 \qquad \gamma ab - c - 2\xi b^2 = 0\,.$$

The solutions are

$$a = \pm\frac{3}{\gamma}\sqrt{\xi c} \qquad \text{and} \qquad b = \pm\frac{1}{2}\sqrt{\frac{3\beta c}{\delta\gamma}} = \pm\frac{1}{2}\sqrt{\frac{c}{\xi}}\,.$$

It requires that $\delta\gamma = 3\beta\xi$.

(1*c*) $\alpha = 2$, $\sigma = 1$, $\tau = 0$. We have the equation system:

$$\beta - 2\delta a^{-1}b - \gamma a^{-1}b + 6\xi a^{-2}b^2 = 0 \qquad \gamma ab - c - 2\xi b^2 = 0\,.$$

We obtain the eight sets of solutions as

$$a_{1,2} = a_{3,4} = \sqrt{\frac{18\xi c}{6\xi\beta - (\delta - \gamma)[2\delta + \gamma \pm \sqrt{(2\delta+\gamma)^2 - 24\beta\xi}]}}$$

$$a_{5,6} = a_{7,8} = -\sqrt{\frac{18\xi c}{6\xi\beta - (\delta - \gamma)[2\delta + \gamma \pm \sqrt{(2\delta+\gamma)^2 - 24\beta\xi}]}}$$

$$b_{1,2} = \frac{2\delta + \gamma + \sqrt{(2\delta+\gamma)^2 - 24\beta\xi}}{4\xi} \times \sqrt{\frac{2\xi c}{6\xi\beta - (\delta - \gamma)[2\delta + \gamma \pm \sqrt{(2\delta+\gamma)^2 - 24\beta\xi}]}}$$

$$b_{3,4} = \frac{2\delta + \gamma - \sqrt{(2\delta+\gamma)^2 - 24\beta\xi}}{4\xi} \times \sqrt{\frac{2\xi c}{6\xi\beta - (\delta - \gamma)[2\delta + \gamma \pm \sqrt{(2\delta+\gamma)^2 - 24\beta\xi}]}}$$

$$b_{5,6} = -\frac{2\delta + \gamma + \sqrt{(2\delta+\gamma)^2 - 24\beta\xi}}{4\xi} \times \sqrt{\frac{2\xi c}{6\xi\beta - (\delta - \gamma)[2\delta + \gamma \pm \sqrt{(2\delta+\gamma)^2 - 24\beta\xi}]}}$$

$$b_{7,8} = -\frac{2\delta + \gamma - \sqrt{(2\delta+\gamma)^2 - 24\beta\xi}}{4\xi} \times \sqrt{\frac{2\xi c}{6\xi\beta - (\delta - \gamma)[2\delta + \gamma \pm \sqrt{(2\delta+\gamma)^2 - 24\beta\xi}]}} .$$

It requires $(2\delta+\gamma)^2 \geqslant 24\beta\xi$ and $6\xi\beta > (\delta - \gamma)[2\delta + \gamma \pm \sqrt{(2\delta+\gamma)^2 - 24\beta\xi}]$ for a and b real numbers. When $(2\delta+\gamma)^2 = 24\beta\xi$, we have

$$a = \pm\sqrt{\frac{72\xi c}{(2\delta+\gamma)(5\gamma - 2\delta)}} \qquad \text{and} \qquad b = \pm\sqrt{\frac{c(2\delta+\gamma)}{2\xi(5\gamma - 2\delta)}} .$$

It requires $2\delta < 5\gamma$.

(1d) $\alpha = 4$, $\sigma = 1$, $\tau = 2$. We have the equation system:

$$\beta - \gamma a^{-1} b = 0 \qquad \gamma ab - 2\delta a^{-1} b + 6\xi a^{-2} b^2 = 0 \qquad c + 2\xi b^2 = 0 .$$

When $\gamma\delta > 3\beta\xi$, the solutions are

$$a = \pm\frac{\sqrt{2(\gamma\delta - 3\beta\xi)}}{\gamma} \qquad b = \pm\frac{\beta}{\gamma^2}\sqrt{2(\gamma\delta - 3\beta\xi)} \qquad c = \frac{-4\beta^2\xi(\gamma\delta - 3\beta\xi)}{\gamma^4} .$$

(2) $\nu = 2$. The equality becomes

$$\beta u^\alpha - c + \delta b u^\sigma\left(\sqrt{a/u} - 3\sqrt{u/a}\right) + 2\gamma b u^{\tau+1}\left(\sqrt{a/u} - \sqrt{u/a}\right) - 4\xi b^2\left(2 - 3u/a\right) \equiv 0 . \tag{50}$$

Similarly, we also only consider a few typical cases.

(2*a*) $\alpha = 0$, $\sigma = -\tau = \frac{1}{2}$. We have the equation system:

$$\beta - c + \delta\sqrt{a}b - 8\xi b^2 + 2\gamma\sqrt{a}b = 0 \qquad 12\xi b/\sqrt{a} - 2\gamma - 3\delta = 0 .$$

The solutions are

$$a = \frac{36(\beta - c)\xi}{(3\delta + 2\gamma)(3\delta - 2\gamma)} \qquad \text{and} \qquad b = \pm\frac{1}{2}\sqrt{\frac{(\beta - c)(3\delta + 2\gamma)}{\xi(3\delta - 2\gamma)}} .$$

It requires $(\beta - c)/(3\delta - 2\gamma) > 0$ for b a real number.
(2*b*) $\alpha = 1$, $\sigma = -\tau = \frac{1}{2}$. We have the equation system:

$$\beta - 3\delta b/\sqrt{a} - 2\gamma b/\sqrt{a} + 12\xi b^2/a = 0 \qquad \delta\sqrt{a}b - c - 8\xi b^2 + 2\gamma\sqrt{a}b = 0 .$$

The four sets of solutions are

$$a_{1,2} = a_{3,4} = \frac{72\xi c}{4\gamma^2 - 9\delta^2 + 48\beta\xi \pm (2\gamma - 3\delta)\sqrt{(2\gamma + 3\delta)^2 - 48\beta\xi}}$$

$$b_{1,2} = \frac{2\gamma + 3\delta \pm \sqrt{(2\gamma + 3\delta)^2 - 48\beta\xi}}{4\xi} \times \sqrt{\frac{2\xi c}{4\gamma^2 - 9\delta^2 + 48\beta\xi \pm (2\gamma - 3\delta)\sqrt{(2\gamma + 3\delta)^2 - 48\beta\xi}}}$$

$$b_{3,4} = -\frac{2\gamma + 3\delta \pm \sqrt{(2\gamma + 3\delta)^2 - 48\beta\xi}}{4\xi} \times \sqrt{\frac{2\xi c}{4\gamma^2 - 9\delta^2 + 48\beta\xi \pm (2\gamma - 3\delta)\sqrt{(2\gamma + 3\delta)^2 - 48\beta\xi}}} .$$

It requires $(2\gamma + 3\delta)^2 \geqslant 48\beta\xi$. When $(2\gamma + 3\delta)^2 = 48\beta\xi$, the solutions have the simple forms:

$$a = \frac{18\xi c}{\gamma(2\gamma + 3\delta)} \qquad \text{and} \qquad b = \pm\frac{1}{4}\sqrt{\frac{c(2\gamma + 3\delta)}{2\gamma\xi}} .$$

(2*c*) $\alpha = 2$, $\sigma = \tau = \frac{1}{2}$. We have the equation system:

$$\beta - 2\gamma b/\sqrt{a} = 0 \qquad 2\gamma\sqrt{a} - 3\delta/\sqrt{a} + 12\xi b/a = 0 \qquad \delta\sqrt{a}b - 8\xi b^2 - c = 0 .$$

The solutions are

$$a = \frac{3(\delta\gamma - 2\beta\xi)}{2\gamma^2} \qquad b = \pm\frac{\beta}{2\gamma^2}\sqrt{\frac{3(\delta\gamma - 2\beta\xi)}{2}} \qquad c = \frac{3\beta(\delta\gamma - 2\beta\xi)(\delta\gamma - 4\beta\xi)}{4\gamma^4} .$$

It requires $\delta\gamma > 2\beta\xi$ for b a real number.
(3) $\nu \neq 1$, $\nu \neq 2$. This case is very complicated in general. We only discuss this case for special conditions: $\beta, \delta, \gamma, \xi \geqslant 0$.

If $\alpha = 2/\nu$, $-\sigma = \tau + 1 = 1/\nu$, we have the equation system:

$$\beta - \gamma\nu b/a^{1/\nu} + \xi(\nu + 1)(\nu + 2)b^2/a^{2/\nu} = 0$$
$$\delta + \xi(\nu - 2)a^{1/\nu}b = 0$$
$$\gamma\nu a^{1/\nu}b - 2\nu^2\xi b^2 - c - \delta(\nu + 1)b/a^{1/\nu} = 0 .$$

The four sets of solutions are

$$a_{1,2} = a_{3,4} = \left\{ \frac{\delta\left[\gamma \mp \sqrt{\gamma^2 - 2\beta\xi(\alpha+1)(\alpha+2)}\right]}{2\beta\xi(\alpha-1)} \right\}^{1/\alpha}$$

$$b_{1,2} = \frac{\alpha}{2\xi}\sqrt{\frac{\delta\left[\gamma \pm \sqrt{\gamma^2 - 2\beta\xi(\alpha+1)(\alpha+2)}\right]}{(\alpha^2-1)(\alpha+2)}}$$

$$b_{3,4} = -\frac{\alpha}{2\xi}\sqrt{\frac{\delta\left[\gamma \pm \sqrt{\gamma^2 - 2\beta\xi(\alpha+1)(\alpha+2)}\right]}{(\alpha^2-1)(\alpha+2)}}$$

$$c_{1,2} = c_{3,4} = \frac{\delta\gamma}{\xi(\alpha-1)} - \frac{\delta(\alpha^2+\alpha+2)}{2\xi(\alpha^2-1)(\alpha+2)}\left[\gamma \pm \sqrt{\gamma^2 - 2\beta\xi(\alpha+1)(\alpha+2)}\right].$$

It requires $\alpha > 1$ and $\gamma^2 \geqslant 2\beta\xi(\alpha+1)(\alpha+2)$. When $\gamma^2 = 2\beta\xi(\alpha+1)(\alpha+2)$, we have

$$a = \left[\frac{\delta(\alpha+1)(\alpha+2)}{\gamma(\alpha-1)}\right]^{1/\alpha} \qquad b = \pm\frac{\alpha}{2\xi}\sqrt{\frac{\delta\gamma}{(\alpha^2-1)(\alpha+2)}} \qquad c = \frac{\gamma\delta(\alpha^2+5\alpha+2)}{2\xi(\alpha^2-1)(\alpha+2)}.$$

Second, we consider applying case 3 to this equation, i.e. assume that the solution can be expressed in the form of (11). We now determine the three parameter, a, b, and n. Substituting the coefficient functions into (12) yields

$$\beta u^\alpha - c + \delta u^\sigma(a + nbu^{n-1}) + \gamma u^\tau(au + bu^n) + \xi\left[a^2 + abn(n+1)u^{n-1} + b^2n(2n-1)u^{2n-2}\right] \equiv 0. \tag{51}$$

For arbitrary values of α, σ, and τ, it is very complicated. Thus, we only consider a special case: $\alpha = \sigma + n - 1 = \tau + n = 2n - 2$. In this case, we have the equation system:

$$\beta + \delta bn + \gamma b + \xi b^2 n(2n-1) = 0$$
$$\delta + \gamma + \xi bn(n+1) = 0$$
$$\xi a^2 - c = 0.$$

When $[2\gamma + \delta(\alpha+2)]^2 = 8\beta\xi(\alpha+1)(\alpha+2)$ and $8(\alpha+1)(\delta+\gamma) = (\alpha+4)[2\gamma+\delta(\alpha+2)]$, the solutions are

$$a = \sqrt{\frac{c}{\xi}} \qquad \text{and} \qquad b = \frac{-4(\delta+\gamma)}{\xi(\alpha+2)(\alpha+4)}.$$

Example 8. We consider a generalized Kuramoto–Sivashinski equation:

$$u_t + \beta u^\alpha u_x + \gamma u^\tau u_{xx} + \delta u_{xxxx} = 0 \tag{52}$$

where α, β, γ, δ, and τ are real numbers. When $\alpha = 1$, $\beta = 1$, and $\tau = 0$, it reduces to the original Kuramoto–Sivashinski [14] equation. The KS equation occurs in the context of modelling chemical reaction–diffusion phenomena, flame-front instability, propagation of long waves on a thin film or on the interface between two viscous fluids. It also serves as a simple model for chaos.

We consider applying case 3 to this equation, i.e. the solution may be expressed in the form of (11). Thus, (12) can be explicitly expressed as

$$\beta u^\alpha - c + \gamma u^\tau(a + bnu^{n-1}) + \delta\left[a^3 + a^2bn(n^2+n+1)u^{n-1} + 3ab^2n^2(2n-1)u^{2n-2} + b^3n(2n-1)(3n-2)u^{3n-3}\right] \equiv 0. \tag{53}$$

This is very complicated in general. We only discuss a special case: $\tau = n - 1$ and $\alpha = 3\tau = 3n - 3$. In this case, we have the equation system:

$$\beta + \delta b^3 n(2n-1)(3n-2) = 0 \qquad \delta a^3 - c = 0$$
$$\gamma + \delta abn(n^2+n+1) = 0 \qquad \gamma + 3\delta abn(2n-1) = 0\,.$$

If and only if when $n = 4$, there are real number solutions for a, b and c, which read

$$a = \frac{\gamma}{6}\left(\frac{5}{49\beta\delta^2}\right)^{1/3} \qquad b = \left(\frac{-\beta}{280\delta}\right)^{1/3} \qquad c = \frac{5\gamma^3}{10584\beta\delta}\,.$$

Example 9. Let us consider the Splading equation:

$$u_t - \beta u_{xx} = K(u^{l+1} + u^{2l+1}) \tag{54}$$

where β and l are real numbers, but $l \neq 0$. (It is a linear differential equation for $l = 0$.)

We can apply case 3 to this equation and obtain the solution as:

$$u(x-ct) = \left\{-\frac{1}{2} - \frac{1}{2}\tanh\left[\pm\frac{l}{2}\sqrt{\frac{-K}{\beta(l+1)}}\left(x \pm \beta\sqrt{\frac{-K}{\beta(l+1)}}\,t\right) + c_0\right]\right\}^{1/l}. \tag{55}$$

Here, it requires $K/\beta(l+1) < 0$ in order for it to be a real function. (Note: parameters $a = b$ and $ab = -K/\beta(l+1) > 0$ in this example.)

Example 10. We now consider a generalized Fitzhugh–Nagumo equation, which reads

$$u_t - \alpha u_{xx} = \beta u(1-u^\delta)(u^\delta - \gamma) \tag{56}$$

where $\alpha, \beta, \delta > 0$ and $\gamma \in [-1, 1)$. Applying case 3 to this equation, we obtain the equation system when $n = \delta + 1$:

$$a(c + a\alpha) - \beta\gamma = 0 \qquad ab\alpha(\delta+2) + bc + \beta(\gamma+1) = 0 \qquad \beta - b^2\alpha(\delta+1) = 0\,.$$

Solving these equations, we finally obtain the two solutions, which read

$$u_1(x-ct) = \left\{\frac{1}{2} + \frac{1}{2}\tanh\left[\mp\frac{\delta}{2}\sqrt{\frac{\beta}{\alpha(\delta+1)}}\left(x \pm (\gamma\delta+\gamma-1)\sqrt{\frac{\alpha\beta}{\delta+1}}\,t + c_0\right)\right]\right\}^{1/\delta} \tag{57}$$

$$u_2(x-ct) = \left\{\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left[\mp\frac{\gamma\delta}{2}\sqrt{\frac{\beta}{\alpha(\delta+1)}}\left(x \mp (\gamma-\delta-1)\sqrt{\frac{\alpha\beta}{\delta+1}}\,t + c_0\right)\right]\right\}^{1/\delta}. \tag{58}$$

Example 11. We now consider a generalized KdV–Burgers equation

$$u_t + \beta u^\alpha u_x - \gamma u^\tau u_{xx} + \delta u_{xxx} = 0 \tag{59}$$

where α, β and τ are positive real numbers. It reduces to the Burgers equation for $\alpha = 1$, $\gamma = 1$, $\tau = 0$, and $\delta = 0$. It also reduces to the KdV equation for $\alpha = 1$, $\gamma = 0$, and $\delta = 1$.

Let us consider applying case 4 to this problem. Substituting the coefficient functions into (16) yields

$$\beta u^\alpha - c - \gamma u^\tau(a_1 + 2a_2 u) + \delta(2a_0a_2 + a_1^2 + 6a_1a_2u + 6a_2^2u^2) \equiv 0\,. \tag{60}$$

For simplicity, we only study the equation for c, α, β, γ, $\delta > 0$. Thus, we consider two cases: (1) $\alpha = \tau = 1$ and (2) $\alpha = 2$, $\tau = 1$.

(1) $\alpha = \tau = 1$. We have the equation system

$$\beta - \gamma a_1 + 6\delta a_1 a_2 = 0 \qquad 2\gamma a_2 - 6\delta a_2^2 = 0 \qquad \delta(2a_0a_2 + a_1^2) - c = 0\,.$$

Some simple algebra yields

$$a_0 = \frac{3\delta}{2\gamma}\left(\frac{c}{\delta} - \frac{\beta^2}{\gamma^2}\right) \qquad a_1 = -\frac{\beta}{\gamma} \qquad a_2 = \frac{\gamma}{3\delta}.$$

Finally, letting $\Delta = 4a_0a_2 - a_1^2 = 2c/\delta - 3\beta^2/\gamma^2$, we have the solution as

$$u(x-ct) = \frac{3\delta\sqrt{\Delta}}{2\gamma}\tan\left[\frac{\sqrt{\Delta}}{2}(x-ct)+c_0\right] + \frac{3\beta\delta}{2\gamma^2} \qquad \text{for} \quad c > \frac{3\delta\beta^2}{2\gamma^2} \tag{61}$$

$$u(x-ct) = -\frac{3\delta\sqrt{-\Delta}}{2\gamma}\tanh\left[\frac{\sqrt{-\Delta}}{2}(x-ct)+c_0\right] + \frac{3\beta\delta}{2\gamma^2} \qquad \text{for} \quad c < \frac{3\delta\beta^2}{2\gamma^2}. \tag{62}$$

(2) $\alpha = 2$, $\tau = 1$. We have the equation system

$$\beta - 2\gamma a_2 + 6\delta a_2^2 = 0 \qquad \gamma a_1 - 6\delta a_1 a_2 = 0 \qquad c - \delta(2a_0a_2 + a_1^2) = 0.$$

If we choose $a_1 = 0$, the solutions are

$$a_0 = \frac{3c}{\gamma \pm \sqrt{\gamma^2 - 6\beta\delta}} \qquad \text{and} \qquad a_2 = \frac{\gamma \pm \sqrt{\gamma^2 - 6\beta\delta}}{6\delta}.$$

If $a_1 \neq 0$, and only if $\gamma^2 = 6\,\beta\,\delta$, we have the solutions as

$$a_0 = \frac{3c}{\gamma} \qquad \text{and} \qquad a_2 = \frac{\gamma}{6\delta}.$$

Finally, letting $\Delta = 4a_0a_2 - a_1^2 = 2c/\delta - a_1^2$ for any real value of $a_1 \neq 0$, we have the solutions as

$$u(x-ct) = \frac{3\delta\sqrt{\Delta}}{\gamma}\tan\left[\frac{\sqrt{\Delta}}{2}(x-ct)+c_0\right] - \frac{3\delta a_1}{\gamma} \qquad \text{for} \quad \Delta > 0 \tag{63}$$

$$u(x-ct) = -\frac{3\delta\sqrt{-\Delta}}{\gamma}\tanh\left[\frac{\sqrt{-\Delta}}{2}(x-ct)+c_0\right] - \frac{3\delta a_1}{\gamma} \qquad \text{for} \quad \Delta < 0 \tag{64}$$

when $a_1 = 0$, the solution is

$$u(x-ct) = \frac{3\delta}{\gamma \pm \sqrt{\gamma^2 - 6\beta\delta}}\sqrt{\frac{2c}{\delta}}\tan\left[\sqrt{\frac{c}{2\delta}}(x-ct)+c_0\right]. \tag{65}$$

Example 12. Let us consider the following nonlinear wave equation:

$$\sqrt{\alpha^2 - u^2}(u_{tt} - u_{xx}) + \beta u u_x = 0. \tag{66}$$

Applying the solution in case 5, it is easy to obtain the harmonic wave solution as

$$u(x-ct) = \alpha\sin\left[\frac{\beta}{c^2-1}(x-ct) - c_0\right]. \tag{67}$$

Example 13. We also consider a generalized KdV equation:

$$u_t + \beta u^\alpha u_x + \gamma u^\tau u_{xxx} = 0 \tag{68}$$

where α, β, γ and τ are positive real numbers.

Let us apply case 5 to this equation, i.e. assume

$$u(x-ct) = a\sin^\nu[b(x-ct)+c_0]. \tag{69}$$

Substituting it into (19) yields

$$\beta u^\alpha - c - \gamma b^2\nu^2 u^\tau\left[1 - \frac{(\nu-1)(\nu-2)}{\nu^2}\left(\frac{a}{u}\right)^{2/\nu}\right] \equiv 0. \tag{70}$$

When $2/\nu = \alpha = \tau$, we obtain the solutions

$$a = \left[\frac{2c}{\beta(\alpha-1)(\alpha-2)}\right]^{1/\alpha} \qquad \text{and} \qquad b = \pm\sqrt{\frac{\alpha^2\beta}{4\gamma}}.$$

These results require that $\alpha \neq 1$ and $\alpha \neq 2$.

Example 14. Consider a modified sine–Gordon equation:

$$u_{xx} + \gamma u_x - u_{tt} = \alpha_1 \sin(\beta u) + \alpha_2 \sin(2\beta u) \tag{71}$$

where α_1, α_2, β, and γ are positive real constants. This equation may be regarded as a higher-order approximation compared with the original sine–Gordon equation. Applying case 7 (let $\nu = 1$) to this equation, it is easy to obtain the solitary wave solutions as

$$u(x-ct) = 2\beta^{-1}\tan^{-1}\left\{\exp\left[\frac{\alpha_1\beta}{\gamma}\left(x \pm \sqrt{\frac{\alpha_1^2\beta - 2\alpha_2\gamma^2}{\alpha_1^2\beta}}\, t - c_0\right)\right]\right\}. \tag{72}$$

Example 15. Similarly, we consider a sine–Gordon-like equation:

$$u_{xx} + \gamma u_x - u_{tt} = \alpha_1 \sin^2(\beta u) + \alpha_2 \sin^3(\beta u)\cos(\beta u) \tag{73}$$

where α_1, α_2, and β are positive real constants. Applying case 8 or case 9 to this problem, it is easy to obtain the solitary wave solution as

$$u(x-ct) = -\beta^{-1}\cot^{-1}\left[\frac{\alpha_1\beta}{\gamma}\left(x \pm \sqrt{\frac{2\alpha_1^2\beta - \alpha_2\gamma^2}{2\alpha_1^2\beta}}\, t - c_0\right)\right]. \tag{74}$$

4. Discussion and conclusions

From the above examples, it is clear to see that the single travelling solution to a nonlinear evolution/wave equation could be obtained by a trial procedure with simple algebraic calculations. It could also be seen that the present method may be generalized for obtaining the multi-travelling solutions, which would be the next step to extend the applications of the technique to achieving the exact solutions to the nonlinear evolution/wave equations.

By introducing a series of ansätze, we have solved several classes of nonlinear partial differential equations, which are used in physical sciences. Through the presentations of various kinds of examples, we may achieve the following concluding remarks.

(1) The present ansatz approach only involves algebraic calculations, which is much easier than the differential and integral derivations, compared with the previous methods.
(2) Using a simple trial procedure, we can determine the condition for the existence of a certain kind of solution.
(3) From the examples given in this paper, one may see that the present method is quite a powerful tool for obtaining exact analytical solutions.

The method used in this paper may be considered as a practical approach to obtaining the travelling (and/or) wave solutions to nonlinear evolution/wave equations. It can in principle be generalized to the investigation of coupled nonlinear evolution/wave equation systems, which is beyond the scope of this paper.

It is worthwhile pointing out that the method used and the solutions obtained in this paper can be generalized to higher-dimensional systems.

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